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LETTER TO THE EDITOR

Painlevé analysis, rational and special solutions of variable coefficient Korteweg–de Vries equations

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Abstract. The Painlevé analysis of two variable coefficient Korteweg–de Vries equations are considered. The Painlevé test for these equations is applied. The condition for the Painlevé property of the equations are found. Some rational and special solutions are presented.

In this letter we want to study the Painlevé property and to find rational solutions for two generalized Korteweg–de Vries equations

$$u_t + \tfrac{3}{2}u^2 u_x + u_{xxx} + u_x f + u f_x + f_{xxx} = 0 \tag{1}$$

and

$$\omega_t + 3\omega\omega_x + \omega_{xxx} + 2\omega g_x + \omega_x g + g_{xxx} = 0 \tag{2}$$

where $f = f(x,t)$ and $g = g(x,t)$ are smooth functions of x and t.

Equations (1) and (2) at $f = g = 0$ are well known Korteweg–de Vries equations [1–2]. Recently (1) and (2) were considered at $f = g = xA(t) + B(t)$ [3, 4], where $A(t)$ and $B(t)$ are arbitrary functions of t. It was shown that these equations have the pseudo potentials of Wahlquist–Estabrook and Lax pairs.

Later we will investigate (1) and (2) on the Painlevé property [5, 6] and find some rational and special solutions.

Firstly we will check these equations on the Painlevé test following [7, 8]. Let us seek solutions of (1) and (2) in the form

$$u = \sum_{j=0}^{\infty} u_j \varphi^{j-1} \tag{3}$$

$$\omega = \sum_{k=0}^{\infty} \omega_k \phi^{k-2} \tag{4}$$

where $\varphi(x,t)$ and $\phi(x,t)$ are the new functions of x and t, and u_j and ω_k are coefficients of expansions (3) and (4).

Singular parts in (3) and (4) are handled taking into account the Painlevé analysis of the usual Korteweg–de Vries equations [5].

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Substituting (3) into (1) shows that resonances occur at $j = -1$, 3 and 4. The compatibility conditions at $j = 3$ and 4 are satisfied identically, therefore (1) possesses the Painlevé test like the simple modified Korteweg–de Vries equation.

Equation (2) can be investigated in a similar manner. This time we have observed resonance at $k = -1$, 4 and 6, which corresponds to resonances usual Korteweg–de Vries equation too. Coefficients ω_4 and ω_σ in the Laurent expansion (4) can be taken as arbitrary functions. Thus we can see that (2) also satisfies the Painlevé test.

Note that the singular manifold equation in both cases takes the form

$$\varphi_t + \varphi_{xxx} - \frac{3\varphi_{xx}^2}{2\varphi_x} + f\varphi_x = 0 \tag{5}$$

but in the second case the replacement $f \Rightarrow g$ and $\varphi \Rightarrow \phi$ in (5) takes place.

Now let us consider the Painlevé property of (1) and (2), which are the sufficiency conditions for integrability of these equations [7, 8].

Let us take equations for the pseudopotentials of Wahlquist–Estabrook in the form [9, 10]

$$q_x = \omega + \frac{q^2}{2} + \lambda \tag{6}$$

$$q_t = -\frac{\partial}{\partial x}(\omega_x + q\omega + g_x + qg) \tag{7}$$

and assume

$$(q_x)_t = (q_t)_x. \tag{8}$$

Then we obtain (2) at

$$\lambda_x = 0 \qquad \lambda_t + \lambda g_x = 0. \tag{9}$$

Thus we obtain that (2) has the pseudo potentials corresponding to (6) and (7) at

$$g(x, t) = xC(t) + D(t) \tag{10}$$

where $C(t)$ and $D(t)$ are arbitrary functions of t.

Using the replacement

$$q = -\frac{2\psi_x}{\psi} \tag{11}$$

in (6) and (7) we can find the Lax pair for (2) at condition (10).

$$\psi_{xx} + \tfrac{1}{2}(\omega + \lambda)\psi = 0 \tag{12}$$

$$\psi_t = \tfrac{1}{2}\psi[\omega_x + C(t)] - \psi_x[\omega + xC(t) + D(t)] \tag{13}$$

$$\lambda_t + \lambda C(t) = 0. \tag{14}$$

We obtain that (2) has the Painlevé property at $g = xC(t) + D(t)$ and (2) is integrable in this case. The method of inverse scattering transform for the case of (2) was presented in [4].

Lax pairs for (1) at $f = xA(t) + B(t)$ in the form $AKNC$ problem [11] were considered in [3]. As one might expect (1) also has the Painlevé property in this case.

Using the truncated expansion [12, 13]

$$u = -\frac{2\varphi_x}{\varphi_0 + \varphi} + \frac{\varphi_{xx}}{\varphi_x} \tag{15}$$

$$\omega = \{\phi; x\} = \frac{\phi_{xxx}}{\phi_x} - \frac{3\phi_{xx}^2}{2\phi_x^2} \tag{16}$$

one can write the following equalities for (1) and (2)

$$\begin{aligned} &u_t + \tfrac{3}{2}u^2 u_x + u_{xxx} + u_x f + u f_x + f_{xxx} \\ &\quad = -\frac{\partial}{\partial x}\left[\left(\frac{2}{\varphi_0 + \varphi} - \frac{1}{\varphi_x}\right)\frac{\partial}{\partial x}\right]\left(\varphi_t + \varphi_{xxx} - \frac{3\varphi_{xx}^2}{2\varphi_x} + f\varphi_x\right) \end{aligned} \tag{17}$$

$$\begin{aligned} &\omega_t + 3\omega\omega_x + \omega_{xxx} + 2\omega g_x + \omega_x g + g_{xxx} \\ &\quad = \left[\left(\frac{2\phi_x}{\phi_0 + \phi} - \frac{\partial}{\partial x} - \frac{\phi_{xx}}{\phi_x}\right)\frac{\partial}{\partial x}\left(\frac{2}{\phi_0 + \phi} - \frac{1}{\phi_x}\right)\frac{\partial}{\partial x}\right] \\ &\qquad \times \left(\phi_t + \phi_{xxx} - \frac{3\phi_{xx}^2}{2\phi_x} + g\phi_x\right). \end{aligned} \tag{18}$$

We can see from (17) and (18) that the Miura transformation [14]

$$\omega = u_x - \frac{u^2}{2} \tag{19}$$

is the link between solutions of (1) and (2) at $f = g$.

Now let us show that (1) and (2) at $f = xA(t) + B(t)$ and $g = xC(t) + D(t)$ have sets of rational solutions.

At first let us consider the following equations

$$z_t + z_x G(\{z, x\}) + f z_x = 0 \tag{20}$$

where $G(\omega)$ smooth functions or operators of $\omega = \{z; x\}$.

Now we need a small theorem.

Theorem. Let (20) have the transformation [15]

$$z_x = \varphi_x^m \qquad m < 0 \tag{21}$$

at $f = 0$, then (20) has transformation in the form

$$z_x = \varphi_x^m \exp((m-1)a(t)) \qquad m < 0 \qquad a(t) = \int A(t)\,\mathrm{d}t \tag{22}$$

at $f = xA(t) + B(t)$.

Proof. Substituting (22) into (20) gives the equality

$$\frac{\partial}{\partial x}(z_t + z_x G(\{z, x\}) + f z_x) = m e^{(m-1)a(t)} \varphi_x^{m-1} \frac{\partial}{\partial x}(\varphi_t + \varphi_x G(\{\varphi, x\}) + f \varphi_x) \tag{23}$$

which proves the theorem.

Equation (5) is the partial case of (20).

It is well known [6–8] that equation (5) at $f = 0$ are invariant under Mobius group

$$z = \frac{l\varphi + m}{n\varphi + k} \qquad lk - mn \neq 0. \tag{24}$$

Obviously equation (5) at $f \neq 0$ is also invariant under transformation (24).

It is also known [15] that (5) at $f = 0$ is invariant under transformation (21) at $m = -1$.

Let us take [15]

$$z = -\frac{1}{\varphi} \tag{25}$$

from (24), then taking into account (22) at $m = -1$ one can obtain the Bäcklund transformation in the form

$$\varphi_{n+1,x} = \frac{\varphi_n^2}{\varphi_{n,x}} e^{-2a(t)}. \tag{26}$$

for (5) at $f = xA(t) + B(t)$.

This transformation can be used for finding rational solutions of (1) and (2) at $f(x, t) = xA(t) + B(t)$.

Let us take the solution of (5)

$$\varphi_0 = x \exp\{-a(t)\} \qquad a(t) = \int A(t)\, dt. \tag{27}$$

Without loss of generality we assume $B = 0$, then (27) is a solution of (5) at $f = xA(t)$.

One can find

$$\varphi_1 = \frac{x^3}{3} e^{-3a} + C_1(t) \tag{28}$$

from transformation (26). Arbitrary function $C_1(t)$ is determined from (5). We find

$$\varphi_1 = \left(\frac{x^3}{3} + 4t\right) e^{-3a(t)} + 12 s(t) \tag{29}$$

where

$$s(t) = \int t A(t) e^{-3a(t)}\, dt. \tag{30}$$

Substituting (29) into (26) gives $\varphi_2(x, t)$.

$$\varphi_2 = \left[\left(\frac{x^5}{45} + \frac{4x^2 t}{3} - \frac{16t^2}{x} + C_2(t)\right) e^{-3a(t)} + \left(4x^2 - \frac{96t}{x}\right) s(t) - \frac{144}{x} e^{3a(t)} s^2(t)\right] e^{-2a(t)} \tag{31}$$

where $s(t)$ is determined by (30) and arbitrary function $C_2(t)$ is determined from (5).

This approach can be continued to find $\varphi_n(x,t)$ $(n \geqslant 3)$, by analogous Weiss [15].

The existence of a set of rational solutions of (5) also say things about the integrability of this equation at $f = xA(t) + B(t)$.

Using rational solutions (27), (29) and (31) one can find the rational solutions of equations (1) and (2) at $f = xA(t) + B(t)$ if we use formulas (15) and (16).

It is well known that the singular manifold equations are convenient for finding special solutions of original equations [16–19].

For example, let us look for the special solutions of (1) taking into account the equation

$$\varphi_t + \varphi_{xxx} - \frac{3\varphi_{xx}^2}{2\varphi_x} + f\varphi_x = E(t)\varphi \tag{32}$$

which is found from (18), where $E(t)$ is arbitrary function of t.

Let us take

$$\varphi(x,t) = r(t)F(\vartheta) \qquad \vartheta = \frac{x}{p(t)}. \tag{33}$$

Substituting (33) into (31) gives the following equality

$$\frac{F_{\vartheta\vartheta\vartheta}}{F_\vartheta} - \frac{3F_{\vartheta\vartheta}^2}{2F_\vartheta^2} = \vartheta p^2 \frac{\mathrm{d}p}{\mathrm{d}t} - fp^2. \tag{34}$$

Assuming in (34)

$$\chi(\vartheta) = \vartheta p^2 \frac{\mathrm{d}p}{\mathrm{d}t} - fp^2 \tag{35}$$

one can obtain a number of solutions of (32) solving the ordinary differential equation

$$\frac{F_{\vartheta\vartheta\vartheta}}{F_\vartheta} - \frac{3F_{\vartheta\vartheta}^2}{2F_\vartheta^2} = \chi(\vartheta). \tag{36}$$

Solutions of (36) can be found for $f(x,t)$ which is determined from (35).

In particular let us take

$$\chi(\vartheta) = 2\vartheta \tag{37}$$

then $f(x,t)$ in (1) takes the form

$$f(x,t) = \frac{x}{p^3}\left(p^2\frac{\mathrm{d}p}{\mathrm{d}t} - 2\right) \tag{38}$$

where $p(t)$ is an arbitrary function of t.

Solution of the equation

$$\frac{F_{\vartheta\vartheta\vartheta}}{F_\vartheta} - \frac{3F_{\vartheta\vartheta}^2}{2F_\vartheta^2} = 2\vartheta \tag{39}$$

has the form

$$F(\vartheta) = C \int_0^{\vartheta} \psi(\xi)^2 \, \mathrm{d}\xi. \tag{40}$$

Using (40) one can obtain solutions of (1) taking into account formula (15)

$$u = -2\psi(\vartheta)^{-2} \left[\varphi_0(t) + \int_0^{\vartheta} \psi(\xi)^{-2} \, \mathrm{d}\xi \right]^{-1} - 2\frac{\mathrm{d}}{\mathrm{d}\vartheta} \ln \psi(\vartheta) \tag{41}$$

where

$$\psi(\vartheta) = C_3 Ai(\vartheta) + C_4 Bi(\vartheta) \tag{42}$$

where C_3 and C_4 are arbitrary constants.

In the case $f = 0$ we find

$$p(t) = (6t + C_5)^{1/3} \tag{43}$$

from (38).

Using (41), (42) and (43) one can obtain the self-similar solution of the modified Korteweg–de Vries equation in this case.

In conclusion let us repeat the results of this letter. We have considered two generalized Korteweg–de Vries equations (1), (2) with variable coefficients and have studied these equations on the Painlevé test, which is the necessary condition for the integrability of equations. We have also shown that the original equations at $f = xA(t) + B(t)$ and $g = xC(t) + D(t)$ have the Painlevé property and consequently satisfy the sufficiency condition of their integrability. For this case we have found non-local Bäcklund transformation, which generalized the Weiss transformation. Using these, transformation sets of rational solutions were obtained. Finally we presented some special solutions of equation (1).

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